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Hence  $V_c - V_s = \frac{1}{5}V$ , since  $V_c - V_s =$  volume of water. Substituting, we get

$$r_1^2(a + x) - x^2(3r - x) = \frac{R^2h}{5}.$$

Since

$$VC = VO - CO \quad \text{and} \quad VO = \frac{r}{\sin \alpha},$$

we get

$$a = r \left( \frac{1}{\sin \alpha} - 1 \right),$$

but since

$$\tan \alpha = \frac{5}{12}, \quad \sin \alpha = \frac{5}{13},$$

therefore

$$a = 2 \left( \frac{13}{5} - 1 \right) = \frac{16}{5}.$$

From triangles  $VEF$  and  $VDN$ , we get

$$r_1 = \frac{(a + x)}{h} R, \quad \text{or} \quad r_1 = \frac{\left( \frac{16}{5} + x \right) \frac{5}{2}}{6} = \frac{4}{3} + \frac{5x}{12}.$$

Substituting these values in the above equation and reducing, we get

$$845x^3 - 3120x^2 + 3840x - 1304 = 0.$$

To solve this equation, let

$$x = y + \frac{108}{169}, \quad \text{or} \quad x = y + \frac{16}{13}.$$

Substituting and reducing, we get

$$169y^3 = -\frac{16^3}{13} + 260.8, \quad \text{or} \quad 13^3y^3 = -705.6.$$

Hence,

$$y = -\frac{\sqrt[3]{705.6}}{13} = -\frac{2}{13} \sqrt[3]{88.2}.$$

Then

$$x = y + \frac{16}{13} = \frac{16 - 2\sqrt[3]{88.2}}{13} = .54595.$$

Hence, the required height  $= a + x = 3.2 + .54595 = 3.74595$  inches.

Excellent solutions were received from NATHAN ALTSHILLER, C. N. SCHMALL, HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, and PAUL CAPRON.

#### CALCULUS.

##### 375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation

$$x^2(a - bx) \frac{d^2y}{dx^2} - 2x(2a - bx) \frac{dy}{dx} + 2(3a - bx)y = 6a^2.$$

##### I. SOLUTION BY H. T. BIGELOW, La Fayette, Indiana.

Let

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad \frac{dy}{dx} = \sum_{n=0}^{\infty} n c_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}.$$

Substituting and collecting terms we have,

$$\sum_{n=0}^{\infty} a(n^2 - 5n + 6) c_n x^n \equiv \sum_{n=0}^{\infty} b(n^2 - 3n + 2) c_n x^{n+1} + 6a^2.$$

Equating the coefficients of each power of  $x$  on the two sides of this equality, we have,

$$\text{for } x^0 \qquad 6ac_0 = 6a^2;$$

$$\text{for } x \qquad 2ac_1 = 2bc_0;$$

$$\text{for } x^2 \qquad 0ac_2 = 0bc_1;$$

$$\text{for } x^3 \qquad 0ac_3 = 0bc_2;$$

and

$$\text{for } x^i, \quad (i^2 - 5i + 6)aci = (i^2 - 5i + 6)bc_{i-1}.$$

Solving these equations;

$$c_0 = a, \quad c_1 = b, \quad c_2 \text{ is arbitrary,} \quad c_3 \text{ is arbitrary,}$$

$$c_4 = \frac{b}{a}c_3, \quad c_5 = \frac{b^2}{a^2}c_3, \quad c_6 = \frac{b^3}{a^3}c_3, \quad \text{etc.}$$

Hence,

$$\begin{aligned} y &= a + bx + c_2x^2 + c_3 \left( x^3 + \frac{b}{a}x^4 + \frac{b^2}{a^2}x^5 + \dots \right) \\ &= a + bx + c_2x^2 + \frac{c_3x^3}{1 - \frac{b}{a}x}, \quad \text{if } |x| < \frac{a}{b}. \end{aligned}$$

Renaming the constants,

$$y = a + bx + Ax^2 + \frac{Bx^3}{a - bx},$$

where  $A$  and  $B$  are arbitrary. By substitution, it is readily verified that this actually is the solution for all values of  $x$ , except of course  $x = \frac{a}{b}$ .

## II. SOLUTION BY W. W. BEMAN, Ann Arbor, Mich.

This problem appears in Article 69 of Forsyth's *Differential Equations* and may be solved by the method there indicated. The resolution into factors may be effected more easily by symbolic methods.

Putting

$$x \frac{d}{dx} \equiv \theta,$$

the equation becomes

$$[a(\theta^2 - 5\theta + 6) - bx(\theta^2 - 3\theta + 2)]y = 6a^2,$$

which may be written in three different ways:

$$[a(\theta - 3) - bx(\theta - 1)](\theta - 2)y = 6a^2, \text{ or}$$

$$(\theta - 3)[a(\theta - 2) - bx(\theta - 1)]y = 6a^2, \text{ or}$$

$$(\theta - 2)[a(\theta - 3) - bx(\theta - 2)]y = 6a^2.$$

Hence, three particular integrals of the equation, with second member 0, are

$$y_1 = x^2, \quad y_2 = \frac{x^2}{a - bx}, \quad y_3 = \frac{x^3}{a - bx},$$

only two of which are independent.

We may now put  $y = y_1v$ , or  $y = y_2w$ , or  $y = y_3z$ , etc.: or  $(\theta - 2)y = v$ , or<sup>1</sup>  $[a(\theta - 2) - bx(\theta - 1)]y = w$ , or  $[a(\theta - 3) - bx(\theta - 2)]y = z$ , etc.

The substitution  $y = y_2w$  leads to the *normal* form, which could have been obtained in the ordinary way,

$$\frac{d^2w}{dx^2} = \frac{6a^2}{x^4}.$$

Hence,

$$w = A + Bx + \frac{a^2}{x^2}.$$

$$\therefore y = \frac{x^2}{a - bx} \left[ A + Bx + \frac{a^2}{x^2} \right].$$

Again, the equation may be written

$$[(\theta - 2)(\theta - 3)(a - bx)]y = 6a^2.$$

$$\therefore (a - bx)y = Ax^2 + Bx^3 + a^2.$$

Also solved by V. M. SPUNAR, ELIJAH SWIFT, A. M. HARDING, and ELMER SCHUYLER.

**376. Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that

$$\frac{1}{z} - \frac{1}{z} (1 - 2xz + z^2)^{1/2} = x + \frac{z}{2} \left( \frac{x^2 - 1}{1 - xz} \right) + \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} (x^2 - 1)^n \left( \frac{z}{1 - xz} \right)^{2n-1}$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

First of all the coefficients of the terms of the infinite series are the same as those in the expansion of  $-(1 - a)^{\frac{1}{2}}$ . These lead us to expand

$$\left\{ 1 - (x^2 - 1) \left( \frac{z}{1 - xz} \right)^2 \right\}^{1/2}$$

by the Binominal theorem, which gives

$$1 - \frac{1}{2}(x^2 - 1) \left( \frac{z}{1 - xz} \right)^2 - \left\{ \frac{1 \cdot 1}{2 \cdot 4} (x^2 - 1)^2 \left( \frac{z}{1 - xz} \right)^4 + \dots \right\}.$$

Noting that the terms in the bracket are the corresponding terms in the infinite series each multiplied by  $z/(1 - xz)$ , we find that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n - 3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} (x^2 - 1)^n \left( \frac{z}{1 - xz} \right)^{2n-1} \\ = - \frac{1 - xz}{z} \left\{ 1 - (x^2 - 1) \left( \frac{z}{1 - xz} \right)^2 \right\}^{1/2} + \frac{1 - xz}{z} - \frac{1}{2}(x^2 - 1) \left( \frac{z}{1 - xz} \right). \end{aligned}$$

<sup>1</sup> These two forms of solution are indicated in the German Edition of Forsyth.